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From population dynamics to resource exploitation: an introduction

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The topic was suggested by a note on Hubbert's curve for predicting decay of resource exploitation [1]. Suggestion came also from interpretation of Hubbert's curve in terms of the Lotka-Volterra (LV) equations by Bardi and Lavacchi [2]. Link with population dynamics was obvious since logistic and LV equations were proposed within the demography science field. The note provides an overview of fundamental models and application results. Details can be found in [3].

Mathematical population dynamics has a history of about two centuries. The first model can be regarded the exponential law [4] of Malthus [1766-1834]. The population volume *x* is written as an increasing exponential *x(t)=exp(bt)x0* where *x0* denotes initial population, *t* denotes time and *b* growth rate. The exponential is the free response of a first-order unstable differential equation *dx/dt=b x(t),* just describing short-term evolution.

In order to better describe long-term evolutions, Malthusian model was refined in XIX century [5] to include mortality rate by Gompertz [1779-1865]. Population volume, written as *x(t)=exp(ln(xmax/x0)(1-exp(-bt))),* asymptotically attains the finite value *xmax*. The differential equation includes the stabilizing factor *ln(xmax/x0)* which is the asymptotic limit of *f(x,n)=(1-(x/xmax)n)/n*.

By inserting *f(x,n)* as a factor in the exponential equation, that is *dx/dt=bxf(x,n),* one obtains the generalized logistic equation ([6], 1959, also known as Richards' equation). The standard logistic equation, for *n=1,* is the following:

Standard logistic equation: *dx/dt=b(1-x(t)/xmax) x(t), x(0)=x0.*

Figure 1 shows three profiles of the normalized logistic rate *(dx/dt)/xmax* with *tmax*=10 time units, the time when the rate attains the maximum value, and *b*=1/(time unit). The unitary

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exponent *n=1* separates two types of skew-symmetric profiles:

1. fast rising profiles with *n<1*,
2. fast decaying profiles with *n>1*.

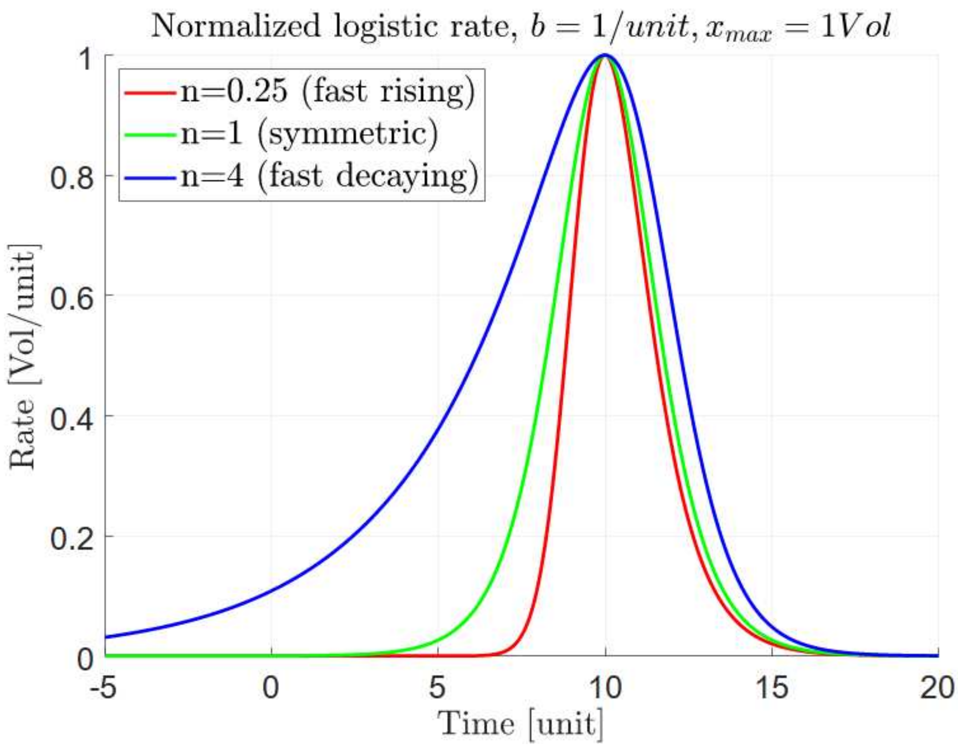

Figure 1. Normalized logistic rate for different n.

The free response *x(t)* of the Richards' equation [3] was proposed in the early XIX century [7] by Verhulst [1804-1849] as a refinement of the exponential model. It is known as the *standard logistic function*, not to be confused with the symmetric logistic rate *y(t)=dx/dt.* The term 'logistic' was created by Verhulst on the Ancient Greek 'logistike' (the art of computing). The population rate *y(t)=dx/dt* grows until *t=tmax* and then decays to zero. The symmetric logistic rate is also known as Hubbert's curve in the field of Earth's resource exploitation, since was proposed by the geologist Hubbert in 1956 [8], thus establishing a link between population dynamics and resource exploitation.

Estimation of the logistic equation parameters *b, xmax, tmax,* where the last parameter replaces *x0,* has been applied to raw data of US crude oil production from 1860 to 2018

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[17]. Figure 2 compares interpolated data (blue) with the estimated profile (red) obtained by composing two Hubbert's curves. Estimation residuals (green) are also shown.

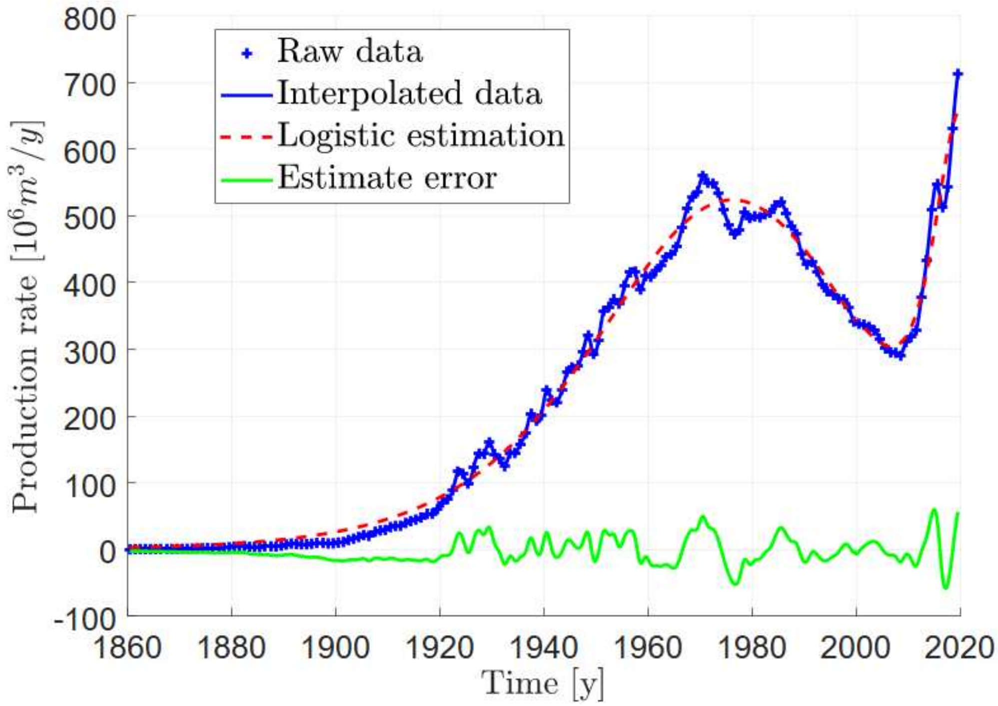

Figure 2. Raw data, estimated Hubbert's curve (logistic rate) and estimation residuals.

Gompertz's and Richards' equations only concern single-species populations. Demographer Lotka (1880-1949) in 1920 [9] and mathematician Volterra (1860-1940) in 1926 [10] proposed two differential equations capable of describing population dynamics of two competing species, the predator with population volume *x* and the prey with volume *z*:

1. predator equation: *dx/dt=-(b-a z(t)) x(t), x(0)=x0,*
2. prey equation: *dz/dt=(c-g x(t)) z(t), z(0)=z0.*

Coefficients *a, b, c* and *g* are positive. Each of the two equations follows the template of the standard logistic equation, but the prey mortality term (negative term) and the predator growth term (positive term) are made extrinsic since they depend on the population of both species, thus accounting for their competition. The free response as shown in Figure 3, left, is a stable

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oscillation where prey and predator increase and decrease their population in quadrature, thus emulating competition in a simplified way. Oscillation amplitude and period in Figure 3, right, just depend on initial conditions, namely on their distance from equilibrium point where both populations remain constant (red lines in Figure 3, right).

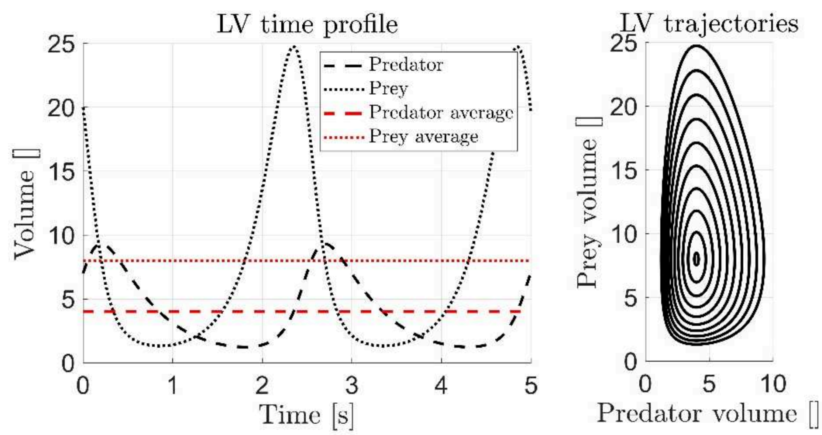

Figure 3. Left: time profiles of the largest trajectory on the right figure. Right: LV trajectories around population average.

Figure 3, left, shows the time profile of the pair predator and prey. Figure 3, right, shows predator and prey trajectories under different initial conditions.

LV equations have been extended to include intrinsic mortality terms and a finite number of species [3]. Generic nonlinear equations were suggested from experiments by Arditi and Ginzburg [11]. Since formulation in 1972 [12] by biologist Maynard Smith [1920-2004] of evolutionary stable strategies - a Nash equilibrium of the game theory which is evolutionary stable, in the sense that only natural selection protects population evolution from small external perturbations - population dynamics has been complemented by game theory [13].

A three-step estimation [3] of the six parameters of LV equations, the four parameters *a, b, c* and *g* and the initial conditions *x0, z0*, has been applied to 'lynx and hare data' collected in the Canadian forests from 1845 to 1933 [18], although recent experimental data in [19] and [20] suggest that LV model is too simple. The recorded data were actually the number of furs caught by Hudson Bay Company, and an underlying assumption is that the number of caught furs was proportional to the actual population of hares and lynxes. The raw data are limited to years from 1900 to 1920 and the unit of the yearly average population volume is kVol=1000 individuals. Raw data (blue circles and crosses) in Figure 4, left, were interpolated with a cubic spline (dashed and solid red curves) to create a smooth profile and compute population

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rate. Estimated profiles are the black curves (dashed and pointwise) indicated in the legend as simulated.

Estimation residuals, normalized by population average, are shown in Figure 4, right. Hare and lynx residuals are close to be each other orthogonal, as proved by a correlation coefficient of -0.24. Estimated profiles in Figure 4, left (black curves), show a significant deviation from raw data during low population periods from year 5 to 13. We can imagine an external cause, like a higher proportion of caught furs with respect to actual population, or a more complex competition dynamics as proved in [19] and [20].

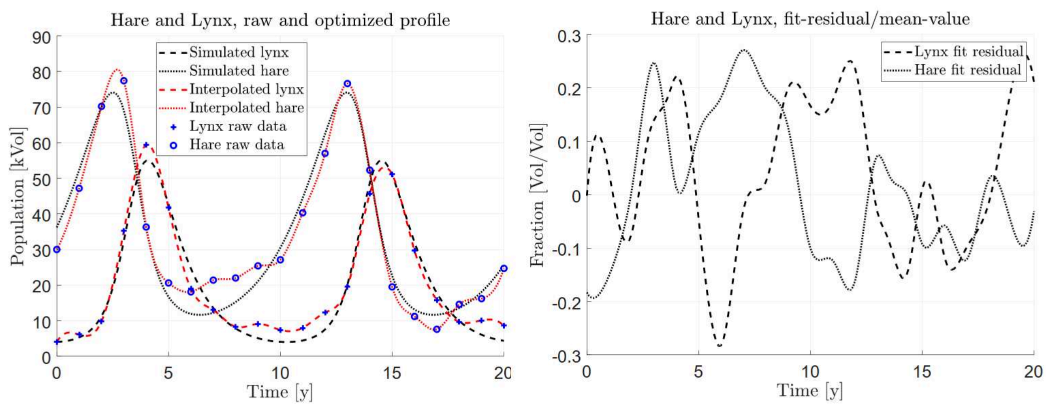

Figure 4. Left: estimated population profile compared to raw data. Right: fractional residuals.

In [2] a limiting form of LV equations has been applied to *resource exploitation* and specifically to justify and generate, under simple assumptions, the bell-shaped Hubbert's curve of the logistic equation as in Figure 1. The key difference is the assumption of two competing species, the resource, playing the prey role, and the human capital, employed in exploitation, playing the predator role. The key model assumption is that resources either cannot reproduce or reproducing rate is negligible (non-renewable resource). Both capital (investment) and resource amount (volume, stock) can be measured in currency, energy or mass units. Equations are the same as LV equations but without the resource self-reproduction factor, implying $c=0$:

1. capital stock equation: $dx/dt=-(b-a\, z(t))\, x(t),\ x(0)=x0,$

2. resource equation: $dz/dt=-g\, x(t)z(t),\ z(0)=z0.$

Two further variables are of interest:

1. the resource production rate $p=-dz/dt==g\, x(t)\, z(t),$

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2. the return on investment (ROI) $q=p/x$ with unit [1/ (time unit)].

Due to $c=0$, LV oscillations disappear which corresponds to infinite period oscillations. It can be shown [3] that the capital time profile follows a bell-shaped curve as the resource production (dashed black and pointwise red curves in Figure 5, left). Instead, resource and ROI monotonically decrease down to a non-zero limit value (black and dashed magenta curves in Figure 5, left). Further, it can be shown that the capital stock is the solution of a linear equation driven by the production rate, which fact is graphically expressed by the phase diagram in Figure 5, right. This favors the design of a simple estimation method, in order to estimate capital obsolescence rate b [1/(time unit)] and transformation coefficient a/g between resource and capital.

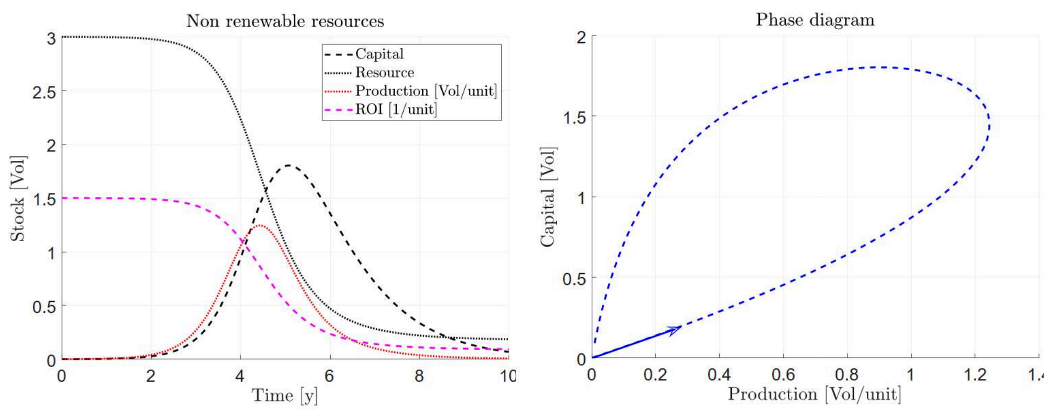

Figure 5. Left: Time profiles of capital stock, resource stock, production and ROI. Right: production-capital phase diagram.

Estimation was applied to California gold rush, [21], [22], with some interesting result. Gold rush (1848-1955) began when gold flakes were found in the American River near Coloma, California. In March 1848, the discovery was announced through the streets of San Francisco by a newspaper publisher, who holding a vial of gold, walked shouting ‘Gold, gold, gold from the American River’. The US-wide spread news attracted from 1848 to 1855 about 300000 people from the rest of United States and abroad. Available data are scarce and uncertain. Gold production is measured in MUSdollar/year=106 USdollar/year. Capital stock is assumed proportional to the number of gold prospectors [kVol]. Production and capital data have been interpolated beginning in 1843 and assuming zero production from 1843 to 1847. The time unit is 1 year.

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No measurement error is known. Capital obsolescence was found to be 0.75 year-1. Resource to capital transformation was estimated about 1.6 prospector per 1000 US dollars, which means that the average prospector produced about 625 US dollars per year. By assuming a price of 0.75 US dollars each gram, the average prospector produced less than 1 kg of gold per year.

Notwithstanding sparse and uncertain data, the interpolated capital profile (blue curve in Figure 6, left) seems capable of fitting the interpolated gold production (blue curve in Figure 6, right) except during the peak production around the 1852 year. Peak production may be the result of improvements in gold-recovery techniques from placer to hydraulic mining. In fact, by assuming the smooth capital curve in Figure 6, left, a production peak higher than the estimated profile should imply that other capitals than prospectors were employed.

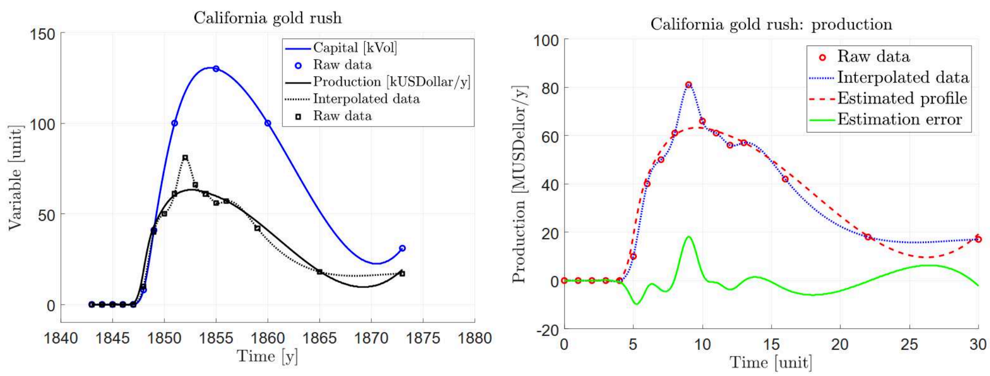

Figure 6. Left: Capital and production raw data and estimated production. Right: production raw and interpolated data, estimated profile and error.

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